 Sample question solutions

Further proofs by mathematical induction

1. Use mathematical induction to prove that for all integers

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

Now, , from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for all integers .

1. Prove that is divisible by if is odd

**Step 1: Base case:** Prove true for

, which is divisible by 13

True for

**Step 2:** Assume that the statement is true for where is an odd integer.

, where is an integer

or and

**Step 3: Inductive step:** Prove the statement to be true for where is an odd integer

Show , where is an integer

Now, from step 2. (the assumption)

 which is divisible by 13 as is an integer.

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any odd number .

1. Prove that is a multiple of if is even

**Step 1: Base case:** Prove true for

, which is divisible by 8

True for

**Step 2:** Assume that the statement is true for where is an even integer.

, where is an integer

or

**Step 3: Inductive step:** Prove the statement to be true for where is an even integer

Show , where is an integer

Now, from step 2 (the assumption)

Now, , as is even.

 which is divisible by 8 as is an integer.

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any even number .

Proofs by mathematical induction involving sigma notation

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

 or

**Step 3: Inductive step:** Prove the statement to be true for

Show

Now, from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any integer .

1. (NESA topic guidance)

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

 or

**Step 3: Inductive step:** Prove the statement to be true for

Show

Now, from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , N, … therefore, true for any integer .

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

 or

**Step 3: Inductive step:** Prove the statement to be true for

Show

Now, from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any integer .

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

 or

**Step 3: Inductive step:** Prove the statement to be true for

Show

Now, from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any integer .

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

 or

**Step 3: Inductive step:** Prove the statement to be true for

Show

Now, from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any integer .

1. (NESA topic guidance)

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

 or

**Step 3: Inductive step:** Prove the statement to be true for

Show

Now, from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any integer .

Prove results using mathematical induction

Inequalities:

1. , for positive integers

**Step 1:** **Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show by proving

 for as increases quicker ( than decreases (

 and

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for all integers .

1. Prove by mathematical induction that: (NESA topic guidance)

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

Compare the centre to the LHS from the assumption. Now

Compare the centre to the RHS from the assumption. Now

and when

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for all integers .

1. Let be a fixed, non-zero number satisfying .

Use the method of mathematical induction to prove that for and hence deduce that for

Prove that for

**Step 1: Base case:** Prove for

 as

**Step 2:** Assume that the statement is true for

i.e.

**Step 3: Inductive step:** Prove the statement to be true for .

 as

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for all integers .

Deduce that for

Algebra:

1. Show that can be written as
2. Using the result in part 1, or otherwise, prove by mathematical induction that, for

 (NESA topic guidance)

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

 using part i)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for all integers .

Divisibility:

1. is divisible by 5 for any positive integer

**Step 1: Base case:** Prove true for

which is divisible by 5 ()

True for

**Step 2:** Assume that the statement is true for

, where is an integer

**Step 3: Inductive step:** Prove the statement to be true for

Show , where is an integer

Now, from step 2 (the assumption)

, which is divisible by 5 as is an integer

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any positive integer .

1. Find the values of when and . Make a conjecture about a number which divides and prove your conjecture by induction. (divisible by 7) (NESA Topic guidance)

When ,

When ,

When ,

When ,

Conjecture: is divisible by 7 when

**Step 1: Base case:** Prove true for

which is divisible by 7

True for

**Step 2:** Assume that the statement is true for

, where is an integer

**Step 3: Inductive step:** Prove the statement to be true for

Show , where is an integer

Now, from step 2 (the assumption)

, which is divisible by 7 as is an integer.

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true any whole number .

Calculus:

1. Prove that for any positive integer ,

**Step 1: Base case:** Prove true for

By first principles

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

Using the product rule

Now, from step 1 and from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any positive integer .

1. Use integration by parts to show that for (NESA topic guidance)

**Step 1: Base case:** Prove true for

Using the double angle formula for cosine:

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

Let

Using integration by parts:

 as required.

1. for when .

**Step 1: Base case:** Prove true for

Apply the product rule with , ,

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

Now, from step 2 (the assumption)

Apply the product rule with , ,

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any positive integer .

1. for when

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

Now, from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any positive integer .

**Probability:**

1. Prove by mathematical induction that:

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

Now, from step 2 (the assumption)

 (splitting out and respectively)

 (Given

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any whole number .

Proof showing

Geometry:

1. Prove that the sum of the exterior angles of an -sided plane convex polygon is 360°,

Let be the exterior angle sum of an -sided polygon, prove

**Step 1: Base case:** Prove true for



 (supplementary angles)

 (supplementary angles)

 (supplementary angles)

 (angle sum of a triangle)

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

Let the polygon be defined by the points , and polygon is defined by (highlighted below)



(from step 2, the assumption)

 (vertically opposite angles)

 (exterior angle of a triangle)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for all integers .

1. If straight lines are drawn in the plane, then the total number of intersections cannot exceed for ,

**Step 1: Base case:** Prove true for

(zero intersections when one straight line is drawn)

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

Let the lines be defined by the

The maximum number of intersections is equal to the number of intersections from lines plus all the intersections created by , which is at most intersections if it intersects all other lines.

Now, from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any positive integer .

1. Suppose we draw on a plane lines in ‘general position’, ie with no three concurrent and no two parallel. Let be the number of regions into which these lines divide the plane, for example in the following diagram:



* 1. By drawing diagrams, find and .











* 1. From these results, make a conjecture about a formula for .

Lead the students to the formula:

Method 1: Using a recursive formula:

Recognise that

 (arithmetic sequence with )

Method 2:

This can be achieved by examining several diagrams in detail such as :



It can be observed every point of intersection is the lowest vertex of some region.

The maximum number of such intersection is (see previous question)

There are 5 additional regions, additional regions.

* 1. Prove this formula by mathematical induction. (NESA topic guidance)

Method 2: Prove, .

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

,

**Step 3: Inductive step:** Prove the statement to be true for

Show

Let the lines be defined by the



The number of regions can be observed as being the number of regions for lines plus an additional regions formed when crosses the existing lines. Note: It can be observed that the additional line creates 5 additional regions when crossing the 4 observable lines.

Now, from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any positive integer .

Method 2: Prove, .

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

Let the lines be defined by the



The number of regions can be observed as being the number of regions for lines plus an additional regions formed when crosses the existing lines. Note: It can be observed that the additional line creates 5 additional regions when crossing the 4 observable lines.

Now, from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any positive integer .

1. For , the sum of the interior angles of a polygon with n vertices is .

Prove

**Step 1: Base case:** Prove true for

 (angle sum of a triangle)

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

Let the polygon be defined by the points



Join



 (base case)

Now, from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any positive integer .

1. An -sided plane convex polygon has diagonals for ,

**Step 1: Base case:** Prove true for

 (diagonals of a triangle)

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

Let the polygon be defined by the points



The number of diagonals is equal to the number of diagonals from a polygon with k sides plus all the diagonals from the point and the diagonal from to .



Diagonals from point join to points , gives () diagonals from and one diagonal from to .

There is a total of additional diagonals.

Now, from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for any positive integer .

Proving first-order recursive formula

1. A sequence is given by the recursive formula for Prove the general formula for the sequence is

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

 (recursive formula)

Now, , from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for all integers .

1. A sequence is given by the recursive formula for Prove the general formula for the sequence is

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

 (recursive formula)

Now, , from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for all integers .

1. A sequence is given by the recursive formula for Prove the general formula for the sequence is

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

(recursive formula)

Now, , from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for all integers .

1. A sequence is given by the recursive formula for Prove the general formula for the sequence is

**Step 1: Base case:** Prove true for

True for

**Step 2:** Assume that the statement is true for

**Step 3: Inductive step:** Prove the statement to be true for

Show

(recursive formula)

Now, , from step 2 (the assumption)

**Step 4: Concluding statement:** If true for , proven true for . Since true for , true for , , … therefore, true for all integers .