 Year 12 Mathematics Extension 2

| MEX- P2 Further proof by mathematical induction | Unit duration |
| --- | --- |
| The topic Proof involves the communication and justification of an argument for a mathematical statement in a clear, concise and precise manner.  A knowledge of proof enables a level of reasoning, justification and communication that is accurate, concise and precise and lays the foundations for understanding the structure of a mathematical argument.  The study of proof is important in developing students’ ability to reason, justify, communicate and critique mathematical arguments and statements necessary for problem-solving and generalising patterns. | 2 to 3 weeks |

| Subtopic focus | Outcomes |
| --- | --- |
| The principal focus of this subtopic is to use the technique of proof by mathematical induction to prove results in series, divisibility, inequality, algebra, probability, calculus and geometry. Students further develop the use of formal mathematical language across various mathematical topics to rigorously and robustly prove the validity of given situations using inductive reasoning. The logical sequence of steps in the proof technique needs to be understood and carefully justified in each application, thus encouraging clear and concise communication which is vital for further study. | A student:   * understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts MEX12-1 * chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings MEX12-2 * applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems MEX12-7 * communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument MEX12-8 |

| Prerequisite knowledge | Assessment strategies |
| --- | --- |
| The material in this topic builds on content from ME-P1: Introduction to Proof by Mathematical Induction from the Mathematics Extension 1 course. MEX-P1 should be taught either concurrently with or prior to the subtopic MEX-P2: Further Proof by Mathematics Induction. | * Staff are encouraged to adopt formative assessment techniques throughout this units as this unit develops high level skills and communication that align to a mastery model of delivery. Peer and self-assessment techniques are also encouraged for students to communicate and interpret solutions effectively. |

All outcomes referred to in this unit come from [Mathematics Extension 2](http://educationstandards.nsw.edu.au/wps/portal/nesa/11-12/stage-6-learning-areas/stage-6-mathematics/mathematics-extension-2-2017) Syllabus  
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Glossary of terms

| Term | Description |
| --- | --- |
| Divisibility | Divisibility is a property when a number can be divided into a given number of groups without any remainders. |
| Inductive step | An inductive step proves that, if the property holds for one natural number k, then it holds for the next natural number k + 1 |
| Initial statement | An initial statement is a statement that holds true for the smallest integer value in a given range. |
| Mathematical induction ⚐ | Mathematical induction is a method of mathematical proof used to prove statements involving the natural numbers.  Also known as proof by induction or inductive proof.  The principle of induction is an axiom and so cannot itself be proven. |
| Proof | Proof is a series of reasons given to prove a statement or a mathematical property. |
| Recursive formula | A recursive formula defines a sequence in which successive terms are expressed as a function of the preceding terms. |

| Lesson sequence | Content | Suggested teaching strategies and resources | Date and initial | Comments, feedback, additional resource used |
| --- | --- | --- | --- | --- |
| Introduction to mathematical induction ME-P1 (1 lesson) | * Revision of ME-P1 Introduction to mathematical induction | **Review of introduction to mathematical induction**   * Re-examine the structure of a proof by mathematical induction   Sample question from ME-P1 unit of work:  Prove by mathematical induction that is divisible by 8 for any positive integer   * + Step 1: Base case. Prove true for the smallest value of n.   Prove true for  , which is divisible by   * + Step 2: Assume true for   , where is an integer   * + Step 3: Inductive step:   Prove true for n=k+1  , where is an integer  Now, , from step 2. (the assumption)  , which is divisible by as is an integer.   * + Step 4: Conclusion: If true for , proven true for . Since true for , true for , , … therefore, true for any positive integer n. |  |  |
| Further proofs by mathematical induction  (1 lesson) | * prove results using mathematical induction where the initial value of is greater than 1, and/or does not increase strictly by 1, for example prove that is a multiple of 8 if is an even positive integer | * **Note**: For solutions to the examples included in this unit, see  mex-p2-sample-question-solutions.DOCX   **Further proofs by mathematical induction**   * Teacher to model completing proofs where:   + The initial value of is greater than 1. Examine how this affects the first step of a proof by mathematical induction.  **Example:** Use mathematical induction to prove that for all integers , (NESA topic guidance)   + does not increase strictly by 1. Examine how this effects the third step of a proof by mathematical induction.  **Example:** Prove that is divisible by if is odd (NESA topic guidance),   + The initial value of n is greater than 1 and n does not increase strictly by 1.  **Example:** Prove that is a multiple of if is even (NESA topic guidance). |  |  |
| Proofs by mathematical induction involving sigma notation (sums) (1 lesson) | * understand and use sigma notation to prove results for sums, for example: | **Proofs by mathematical induction involving sigma notation (sums)**   * Students to be introduced to sigma notation using simple examples and students practise converting from sigma notation to an expanded series and vice versa. * Students look at rules associated with using sigma notation.   **Note**: While this is not explicitly stated in the syllabus it will assist students if formulating proofs by mathematical induction.   * Students could be introduced to mathematical induction involving sigma notation by initially revisiting series problems they did in Extension 1 but now written using sigma notation. * Introduce the ideas of “rolling out” using sigma notation, **e.g.** , and “rolling up”, i.e.) * Sample questions: Prove by mathematical induction that:   + (NESA topic guidance)   + (NESA topic guidance) |  |  |
| Prove results using mathematical induction (1 lesson) | * Understand and prove results using mathematical induction, including inequalities and results in algebra, calculus, probability and geometry. For example: Critical and creative thinking icon   + prove inequality results, e.g. , for positive integers   + prove divisibility results, e.g. is divisible by 5 for any positive integer   + prove results in calculus, e.g. prove that for any positive integer ,   + prove results related to probability, e.g. the binomial theorem:   + prove geometric results, e.g. prove that the sum of the exterior angles of an n-sided plane convex polygon is 360° | **Prove results using mathematical induction**  Students are to understand and prove results using mathematical induction, including but not limited to mathematical results pertaining to:   * **Inequalities**. Inequality statements are difficult to communicate and often difficult to interpret initially. Communication and interpretation techniques require explicit teaching. Inequality statements are restricted to a LHS and RHS, like equations. Inequality statements may contain multiple parts linked by inequality or equality operators  i.e. * This statement may only be interpreted as the inequality operators point in the same direction and may be interpreted as by linking the first part to the last part. * For inequality statements like students may like to adopt the techniques where all terms are collected on one side of the statement and tested to be greater than or less than zero, i.e.) test   Examples:   * + , for positive integers   + Prove by mathematical induction that: (NESA topic guidance)   + Let be a fixed, non-zero number satisfying .   Use the method of mathematical induction to prove that for and hence deduce that for   * **Algebra.** Example:   + Show that can be written as   + Using the result in part (i), or otherwise, prove by mathematical induction that, for ,. (NESA topic guidance) * **Divisibility.** Examples:   + is divisible by 5 for any positive integer   + Find the values of when and . Make a conjecture about a number which divides and prove your conjecture by induction. (divisible by 7) (NESA Topic guidance) * **Calculus.** Examples:   + Prove that for any positive integer ,   + Use integration by parts to show that for (NESA topic guidance)   + for when .   + for when * **Probability.** Example:   + Prove by mathematical induction that: * **Geometry.** Students are encouraged to draw diagrams when proving geometric results. Examples:   + Prove that the sum of the exterior angles of an n-sided plane convex polygon is 360°   + If straight lines are drawn in the plane, then the total number of intersections cannot exceed for   + Suppose we draw on a plane lines in ‘general position’, i.e. with no three concurrent and no two parallel. Let be the number of regions into which these lines divide the plane, for example in the following diagram:   Three lines - none of which are concurrent. Each section between or next to lines are number, from 1 to 7.   * 1. By drawing diagrams, find ,,, and .   2. From these results, make a conjecture about a formula for .   3. Prove this formula by mathematical induction.  (NESA topic guidance)   + For , the sum of the interior angles of a polygon with n vertices is °.   + An -sided plane convex polygon has diagonals for |  |  |
| Proving first-order recursive formula.  (1 lesson) | * Use mathematical induction to prove first-order recursive formulae | **Proving first-order recursive formula**   * A recursive formula defines a sequence in which successive terms are expressed as a function of the preceding terms.   A first-order recursive formula is a sequence in when a term is a function of the immediately preceding term. i.e. The 5th term will depend on the value of the 4th term.  A second-order recursive formula is a sequence in when a term is a function of the two preceding terms. i.e. The 5th term will depend on the value of the 4th and 3rd terms.  Students are to prove first-order recursive formulae using mathematical induction.   * Introduction: consider a recurrence relation, let the first term be 1 and to get the next term we double it and then add one or  1   Students can list the first six terms of the sequence.  , , , ,  Key questions:   * + Why do we need a general formula where is a function of ? i.e   + How would you find the thousandth term?   By examining these terms, students are to hypothesise a general formula relating each term to its position in the sequence, , i.e.  We can now use mathematical induction to prove this is the correct general formula for the recursive relationship. (example 1)  Examples:   * + A sequence is given by the recursive formula  , for . Prove the general formula for the sequence is   + A sequence is given by the recursive formula  , for . Prove the general formula for the sequence is   + A sequence is given by the recursive formula  , for . Prove the general formula for the sequence is   + A sequence is given by the recursive formula  , for . Prove the general formula for the sequence is |  |  |

Reflection and evaluation

Please include feedback about the engagement of the students and the difficulty of the content included in this section. You may also refer to the sequencing of the lessons and the placement of the topic within the scope and sequence. All Information and communication technologies (ICT), literacy, numeracy and group activities should be recorded in the ‘Comments, feedback, additional resources used’ section.