 Year 12 Mathematics Extension 2

| MEX-C1 Further integration | Unit duration |
| --- | --- |
| The topic Calculus involves the study of how things change and provides a framework for developing quantitative models of change and deducing their consequences. This topic involves the development of a broader range of techniques and strategies to solve complex problems related to differential equations and integration.The study of calculus is important in developing students’ knowledge, understanding and capacity to operate with and model change situations involving a variety of functions, use algebraic and graphical techniques to describe and solve problems and to predict future outcomes with relevance to, for example Chemistry, Physics and the construction industry. | 4 to 5 weeks |

| Subtopic focus | Outcomes |
| --- | --- |
| The principal focus of this subtopic is extending students’ knowledge, skills and understanding to a broader range of integration techniques, such as integration of rational functions, integration using partial fractions and integration by parts.Students develop an awareness and understanding of the interconnectedness of topics across the syllabus, and the fluency that can be obtained in the use of calculus techniques. Later studies in mathematics place prime importance on familiarity and confidence in a variety of calculus techniques as these are used in many different fields. | A student:* understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts MEX12-1
* applies techniques of integration to structured and unstructured problems MEX12-5
* applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems MEX12-7
* communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument MEX12-8
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| Prerequisite knowledge | Assessment strategies |
| --- | --- |
| The material in this topic builds on content from the Year 12 Mathematics topic of MA-C4 Integral calculus and the Year 12 Mathematics Extension 1 topic of ME-C2 Further calculus skills. | * Staff should try to adopt formative assessment techniques throughout this unit to develop mastery. These may include using mini-whiteboards, or similar, to provide instantaneous feedback to techniques like competing the square, partial fractions or division transformation. Most formative assessment techniques will include providing feedback to students’ solutions for integration questions.
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All outcomes referred to in this unit come from [Mathematics Extension 2](http://www.educationstandards.nsw.edu.au/wps/portal/nesa/11-12/stage-6-learning-areas/stage-6-mathematics/mathematics-extension-2-2017) Syllabus
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Glossary of terms

| Term | Description |
| --- | --- |
| Distinct factors | Factors which are different from one another |
| Integrand | An integrand is a function that is to be integrated. |
| Rational function | An algebraic fraction such that the numerator and denominator are polynomials |
| Recursive formula | A recursive formula defines a sequence in which successive terms are expressed as a function of the preceding terms. |
| Substitution | Substitution is a technique that defines and uses a parameter to convert an expression or equation, without changing the integrity of the expression or equation. |

| Lesson sequence | ContentStudents learn to: | Suggested teaching strategies and resources  | Date and initial | Comments, feedback, additional resources used |
| --- | --- | --- | --- | --- |
| Integration by substitution (1 or 2 lessons) | * find and evaluate indefinite and definite integrals using the method of integration by substitution, where the substitution may or may not be given Critical and creative thinking icon
 | **Integration by substitution*** At a Mathematics Extension 1 level (ME-C2), students only needed to use this technique when given expression to substitute; however, Mathematics Extension 2 students need to identify and define the expression to substitute.
* The technique of substitution uses the property

 or similarly where In simple cases, students recognise it is appropriate to apply this method when taking an integral of a function multiplied by its derivative.* The expression for substitution can be identified by recognising and in the integrand, then defining . Note that the parameter is generally used but this is not mandatory.
* The technique of substitution is used to make complicated integrals simpler to solve. With indefinite integrals, students must remember to use the substitution in reverse after calculations to express the final integrand in terms of .
* Students need to be exposed to examples of the type:
	+ using the substitution .
	+ using the substitution .
	+ , where .
	+ Find .
* Students need to use typical substitutions such as:
	+ using the substitution
	+ Use the substitution to show that .
* Students need to recognise when to use the Pythagorean results when substituting.

This method enables us to obtain a trigonometric function which can be integrated.Pythagorean result:To use these results, if the denominator contains:* + use the substitution:
	+ use the substitution: .

Note: Questions may be more complex case: use the substitution: . Sample Question:* + By completing the square, find:

Online integral calculators can show solutions with steps. e.g. [Integral-calculator.com](https://www.integral-calculator.com/)**Resources:** mex-c1-sample-questions-solutions.DOCXNote: mex-c1-sample-questions-solutions.DOCX contains worked solutions for sample questions throughout this unit of work. These include NESA sample questions from their topic guidance. |  |  |
| Integration of rational functions with a quadratic denominator(1 lesson) | * integrate rational functions involving a quadratic denominator by completing the square or otherwise Critical and creative thinking icon
 | **Integration of rational functions with a quadratic denominator*** Define a rational function: An algebraic fraction such that the numerator and denominator are polynomials
* Integrate rational functions with a quadratic denominator which, through simple algebra, can be changed into a form that can be integrated.
	+

**The division transformation method** can be applied when the degree of the numerator is the degree of the denominator. To use this method, complete the division before integrating.* Integrate functions with a quadratic denominator that can be factorised using completing the square or otherwise.

Completing the square sample questions:Other methods of factorising may include perfect squares, sum or difference of two squares or factorisation not involving a special result.Note: When a quadratic denominator is factorised, the result may require the use of partial fractions to integrate.**Resource:** mex-c1-sample-questions-solutions.DOCX |  |  |
| Integration using partial fractions(1 lesson) | * decompose rational functions whose denominators have simple linear or quadratic factors, or a combination of both, into partial fractions Critical and creative thinking icon
* use partial fractions to integrate functions Critical and creative thinking icon
 | **Integration using partial fractions**This method is used to decompose or deconstruct a rational function into the sum of several fractions.To apply this method:* The degree of the denominator is larger than the degree of the numerator.

Note: If the degree of the numerator is the degree of the denominator, use the division transformation method first.* The denominator is factorised into a product of distinct factors
* The function is decomposed into the sum of multiple fractions. In general:
* Using the addition of fractions technique, the following identity can be generated

Note that the equivalence sign is used here because this holds true for all values of , including complex numbers. It is not an equation but an identity.* For the the numerators must match, as the denominators match.
* At this point two techniques are available to students
	+ Matching coefficients and constants on both sides of the identity forms simultaneous equations which can be solved. This is time consuming.
	+ Choose appropriate values for to substitute into the identity and eliminate one of the terms. For example if (think then the term is eliminated leaving an expression including the coefficient only. Therefore the value of can be identified. Similarly, choosing a value of eliminates the term .
* Integrate the resulting fractions.

Students need to examine functions which can be expressed as a product of distinct linear factors, a distinct quadratic and linear factor or two distinct quadratic factors.Format of fractions for more complex denominators:* A quadratic factor such as
* A repeated factor such as

[Symbolab partial fraction](https://www.symbolab.com/solver/partial-fractions-calculator) calculator will decompose a function into partial fractions.Sample examples:* +
	+ The expression can be written as where , and are real numbers.

(a) Find , and . (b) Hence find .* + Decompose into partial fractions and hence show that
	+ Decompose into partial fractions and hence show that .

**Resource:** mex-c1-sample-questions-solutions.DOCX |  |  |
| Integration by parts(2 lessons) | * evaluate integrals using the method of integration by parts (ACMSM123)
	+ develop the method for integration by parts, expressed as or

, where and are both functions of Critical and creative thinking icon | **Integration by parts*** Development of integration by parts:

Consider the product rule:Integrate both sides with respect to .Rearranging we obtain:Substituting and we obtain* Integration by parts can be applied when integrating the product of two functions (where integration by substitution cannot be used)

The function should be chosen so that it can be easily integrated. For example, in the first sample question below, *,* let , as cannot be easily integrated.Sample questions:* + where n is an integer e.g.
	+ Use integration by parts to find .
* Integration by parts can be applied to integrate a function which cannot be otherwise integrated by letting

Sample questions:* Harder questions involving integration by parts may need **multiple applications.**

In this scenario, the original function will reappear allowing the development of a solution.For example, questions may contain sin or cos as the original function will reappear after integrating or differentiating twice.Assuming both functions can be easily integrated or differentiated, either can be chosen as and .Questions may be leading or require students to recognise the need to apply integration by parts multiple times.* + Using two applications of integration by parts, evaluate
	+ .

**Resource:** mex-c1-sample-questions-solutions.DOCX |  |  |
| Recurrence relationships(1 lesson) | * derive and use recurrence relationships
 | **Recurrence relationships*** Recurrence relationship: A sequence where each term is defined by a preceding term.
* Integration by parts should be extended to recurrence relations with one integer parameter, such as or .

Sample questions:* +
* Recurrence relations may require the decomposition of the function.

In the example below, students can take out and use the Pythagorean result, to obtain two components, one which can be integrated and another which is a recurrence of the function.* + For let .

Show that for , . Hence or otherwise calculate .* **Note:** Relations such as , which involve more than one integer parameter, are excluded. (Source: NESA topic guidance)

**Resource:** mex-c1-sample-questions-solutions.DOCX |  |  |
| Applications of integration (1 lesson) | * apply these techniques of integration to practical and theoretical situations **AAM** Critical and creative thinking icon Information and communication technology capability icon
 | **Applications of integration*** The skills developed in this unit will be applied with the unit MEX-M1 Applications of Calculus to Mechanics.
* Integration by parts: Students may like to enrich this topic by investigating the Surge model, , and/or the Power Surge Model, , which model the rate of release of medications into the blood stream but have recently been adopted to model the rate of sales during advertising campaigns for products or the rate of hits for online gaming apps.
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Reflection and evaluation

Please include feedback about the engagement of the students and the difficulty of the content included in this section. You may also refer to the sequencing of the lessons and the placement of the topic within the scope and sequence. All ICT, literacy, numeracy and group activities should be recorded in Comments, Feedback, Additional Resources Used sections.