 Year 11 mathematics extension 1

| ME-C1 Rates of change | Unit duration |
| --- | --- |
| The topic Calculus involves the study of how things change and provides a framework for developing quantitative models of change and deducing their consequences. It involves the development of the connections between rates of change and related rates of change, the derivatives of functions and the manipulative skills necessary for the effective use of differential calculus.The study of calculus is important in developing students’ knowledge and understanding of related rates of change and developing the capacity to operate with and model situations involving change, using algebraic and graphical techniques to describe and solve problems and to predict outcomes with relevance to, for example the physical, natural and medical sciences, commerce and the construction industry. | 4 weeks |

| Subtopic focus | Outcomes |
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| The principal focus of this subtopic is for students to solve problems involving the chain rule and differentiation of the exponential function, and understand how these concepts can be applied to the physical and natural sciences.Students develop the ability to study motion problems in an abstract situation, which may in later studies be applied to large and small mechanical systems, from aeroplanes and satellites to miniature robotics. Students also study the mathematics of exponential growth and decay, two fundamental processes in the natural environment. | A student:* uses algebraic and graphical concepts in the modelling and solving of problems involving functions and their inverses ME11-1
* applies understanding of the concept of a derivative in the solution of problems, including rates of change, exponential growth and decay and related rates of change ME11-4
* uses appropriate technology to investigate, organise and interpret information to solve problems in a range of contexts ME11-6
* communicates making comprehensive use of mathematical language, notation, diagrams and graphs ME11-7
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| Prerequisite knowledge | Assessment strategies |
| --- | --- |
| The material in this topic builds on content from MA-E1 Logarithms and Exponentials and MA-C1 Introduction to Differentiation. | * **Summative Assessment:** Can speed be measured perfectly? – An investigative task in which students apply their understanding of calculus techniques to model motion in a straight line. Students will deepen their understanding by researching commercial applications of regression analysis and polynomial models. (Assessment of Learning)
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All outcomes referred to in this unit come from [Mathematics Extension 1](http://educationstandards.nsw.edu.au/wps/portal/nesa/11-12/stage-6-learning-areas/stage-6-mathematics/mathematics-extension-1-2017) Syllabus
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Glossary of terms

| Term | Description |
| --- | --- |
| direct variation | Two variables are in direct variation if one is a constant multiple of the other. This can be represented by the equation , where is the constant of variation (or proportion). Also known as direct proportion, it produces a linear graph through the origin. |
| exponential growth and decay | Exponential growth occurs when the rate of change of a mathematical function is positive and proportional to the function’s current value. Exponential decay occurs in the same way when the growth rate is negative. |
| instantaneous rate of change | The instantaneous rate of change is the rate of change at a particular moment. For a differentiable function, the instantaneous rate of change at a point is the same as the gradient of the tangent to the curve at that point. This is defined to be the value of the derivative at that particular point. |
| rate of change | A rate of change of a function is where is the change in and is the corresponding change in . |
| sketch | A sketch is an approximate representation of a graph, including labelled axes, intercepts and any other important relevant features. Compared to the corresponding graph, a sketch should be recognisably similar but does not need to be precise. |

Lesson sequence

| Lesson sequence | ContentStudents learn to: | Suggested teaching strategies and resources  | Date and initial | Comments, feedback, additional resources used |
| --- | --- | --- | --- | --- |
| Introducing constant v variable rates of change(2 lessons) | **C1.1 Rates of change with respect to time*** describe the rate of change of a physical quantity with respect to time as a derivative Literacy icon
	+ use appropriate language to describe rates of change, for example ‘at rest’, ‘initially’, ‘change of direction’ and ‘increasing at an increasing rate’
 | **Assumed knowledge*** Students need to build on the ideas introduced during stages 4 and 5, with the following key ideas established.
	+ Distinguish the difference between constant and variable rates from graphs
	+ Discuss how the average speed can be calculated as a constant speed
	+ Introduce the idea of a rate of change between two quantities as the gradient of the curve showing the relationship between them

Investigating various rates of change* Identify prior learning by introducing a simple distance-time graph with constant speeds. Discuss the speeds at various stages:
	+ Which stage of the graph shows the greatest speed?
	+ When does it show that the object is stationary?
	+ How does the graph show constant speed?
* Establish the idea that the gradient defines the speed and that a gradient of zero means the object is stationary.
* Represent the gradient equation as , the change in , over , the change in .
* Direct this representation to variable instantaneous rates and use this representation for infinitely small changes in , , over infinitely small changes in , which through the application of limits becomes . Maintain the idea of a fraction involving infinitely small changes as this idea supports the idea of the chain rule, which will be used later in this unit.
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| Applying calculus to model real life scenarios(2 lessons) | * describe the rate of change of a physical quantity with respect to time as a derivative Literacy icon
* investigate examples where the rate of change of some aspect of a given object with respect to time can be modelled using derivatives **AAM**
* Find and interpret the derivative , given a function in the form , for the amount of a physical quantity present at time
* describe the rate of change with respect to time of the displacement of a particle moving along the -axis as a derivative or
	+ Describe the rate of change with respect to time of the velocity of a particle moving along the -axis as a derivative or .
 | Investigating real life examples of modelling using calculus* Students need to be exposed to various situations in the physical, chemical and biological worlds where one quantity changes variably over time. Examples may include motion of an object or mobile phone charge over time; blood sugar levels over time or heart rates before, during and after exercise; the growth of mould over time.
* Class activities:
	+ Deliver this YouTube [air powered bottle rocket](https://www.youtube.com/watch?v=R625vwA4jpQ)(duration 4:49)

Ask students to sketch a distance time graph of the entire journey. Discuss stages of acceleration and deceleration and how that will be represented on the distance-time graph.* + Deliver this YouTube [strobe of a falling ball](https://www.youtube.com/watch?v=xQ4znShlK5A) (duration 0:41).

Assuming the strobe lighting flashes every 0.1s, record the distance fallen against time for this experiment. Create a scatterplot of your results.Applying calculus techniques to model real life scenarios* Staff may like to run the following activities:
	+ An introduction to displacement, velocity and acceleration through an experiment for riding a scooter down a small slope.

**Resource**: introduction-to-displacement-velocity-and-acceleration.DOCX and results-from-downhill-scooter-experiment.GGB* + A guide to fitting polynomials to a data points in Geogebra. **Resource**: user-guide-to-fitting-polynomials-to-scatterplots.DOCX
* Examples of equations that could be used include:
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| Introduction to exponential growth and decay(2 lessons) | **C1.2: Exponential growth and decay*** construct, analyse and manipulate an exponential model of the form to solve a practical growth or decay problem in various contexts (for example population growth, radioactive decay or depreciation) AAM Sustainability icon  Information and communication technology capability icon Civics and citizenship icon
 | Developing the continuous exponential model* Students need to be introduced to situations and concrete experiences that exhibit exponential growth and decay. Links need to be established with, and developed on, the discrete exponential models for compound interest and depreciation. Students need to be introduced to the continuous exponential model in the form and how this relates to the discrete model.
* Students may like to deepen their understanding by investigating that Euler’s number is derived from the compound interest formula.
* Staff may like to run the following activities:
	+ Modelling exponential growth using dice. This activity is a simulation of exponential growth using the outcomes of the dice. **Resource**: modelling-exponential-growth-using-dice.DOCX
	+ Modelling exponential decay using dice. This activity is a simulation of exponential decay using the outcomes of the dice. **Resource**: modelling-exponential-decay-using-dice.DOCX
	+ Modelling radioactive decay using coins. This activity uses coins, Skittles or M&Ms to simulate the decay of atoms in a material experiencing radioactive decay. **Resource**: modelling-the-decay-of-a-Radioactive-Nucleus.DOCX
	+ Modelling exponential growth through paper folding. **Resource**: exponential-growth-and-paper-folding.DOCX
	+ Legend of Paal Payasam, Exponential Growth and Rice on a Chess Board. **Resource**: exponential-growth-and-rice-on-a-chess-board.DOCX
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| Establishing a first order differential model to represent exponential growth(1 lesson) | * construct, analyse and manipulate an exponential model of the form to solve a practical growth or decay problem in various contexts (for example population growth, radioactive decay or depreciation) AAM Sustainability icon  Information and communication technology capability icon Civics and citizenship icon
	+ establish the simple growth model,
* where is the size of the physical quantity, at time and is the growth constant
* construct, analyse and manipulate an exponential model of the form to solve a practical growth or decay problem in various contexts (for example population growth, radioactive decay or depreciation) AAM Sustainability icon  Information and communication technology capability icon Civics and citizenship icon
	+ verify (by substitution) that the function satisfies the relationship , with being the initial value of
 | Deriving the first order differential equation for exponential growth* Students need to determine the exponential growth model as a first order differential equation. Differential equations will be elaborated upon in the applications of calculus unit in year 12; however, this may be an opportunity to introduce this simple case. Students need to be able to interpret the growth constant; firstly in terms of sign and how this impacts on the derivative, **i.e.** when k is positive then the derivative is increasing, which links to exponential growth, and vice versa when k is negative; and secondly, in terms of value
* Staff may like to run the following activities
	+ Establishing the simple exponential model. **Resource**: establish-the-simple-exponential-model.DOCX
	+ Geogebra applet establishing the simple exponential model. **Resource**: geogebra-applet–establish-the-simple-exponential-model.GGB
	+ What does k mean? **Resource**: establish-the-meaning-of-k.DOCX
* Staff can use the [additional background reading](http://www.amsi.org.au/ESA_Senior_Years/SeniorTopic3/3_md/SeniorTopic3e.html#content_5) on continuous growth (exponential) and discrete growth such as interest.

Verify the first order differential equation for exponential growth* Students need to determine the first order differential equation

(Note: is the initial value of )Optional: Integrate to obtain Refer to [amsi](http://www.amsi.org.au/ESA_Senior_Years/SeniorTopic3/3_md/SeniorTopic3e.html#content_1) for modelled proof using integration. |       |       |
| Identify and interpret exponential curves(1 lesson) | * construct, analyse and manipulate an exponential model of the form to solve a practical growth or decay problem in various contexts (for example population growth, radioactive decay or depreciation) AAM Sustainability icon  Information and communication technology capability icon Civics and citizenship icon
	+ sketch the curve for positive and negative values of
	+ recognise that this model states that the rate of change of a quantity varies directly with the size of the quantity at any instant
 | Identify and interpret exponential curves* Students need to sketch and identify curves for exponential growth and decay.
* Staff may like to investigate the curves for exponential growth and decay using the following activity with accompanying key questions:
	+ Geogebra applet to investigate curves of exponential growth and decay. **Resource**: geogebra-applet–graphing-exponential-growth-and-decay.GGB
* Examine a range of graphs as a class (or individually) to consider the following (some sample responses provided):
	+ What effect does the value of have on the curve?

When is positive the graph represent exponential growth.When is negative the graph represent exponential decay.A larger (absolute) value is graphically shown through a faster rate of growth or decay (steeper graph).* + How is A represented on the graph?

It is the intercept* + When sketching the graph of , what information should be included?
	+ This resource outlines the links between discrete quantities used in the Compound Interest model and the continuous exponential model, **Resource**: another-way-to-determine-k.DOCX
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| Applying simple exponential models to real life situations(1 or 2 lessons) | * solve problems involving situations that can be modelled using the exponential model or the modified exponential model and sketch graphs appropriate to such problems AAM Critical and creative thinking icon
 | Applying simple models to real life situations* Staff may like to run the following activities:
	+ Matching activity: Students need to use sets of conditions to link the population model and the simple exponential model. **Resource**: matching-activity-for-simple-exponential-growth-and-decay.DOCX
	+ Another way of determining k. This resource outlines the links between discrete quantities used in the Compound Interest model and the continuous exponential model. **Resource**: another-way-to-determine-k.DOCX
	+ The following comprehensive resource contains class activities that provides students with opportunities to apply models of exponential growth and decay, including
	+ The world’s population
	+ Amazon’s net sales
	+ Google’s worldwide revenue
	+ Growth of rabbit population
	+ Decay of Carbon-14
	+ Nesting Leatherback Turtles
	+ Stopping distance of a car
	+ A bouncing ball

**Resource**: applications-of-exponential-growth-and-decay.DOCX |       |       |
| Introducing the modified exponential model(1 lesson) | * establish the modified exponential model, for dealing with problems such as ‘Newton’s Law of Cooling’ or an ecosystem with a natural ‘carrying capacity’ AAM Sustainability icon  Information and communication technology capability icon Civics and citizenship icon
* establish the modified exponential model, for dealing with problems such as ‘Newton’s Law of Cooling’ or an ecosystem with a natural ‘carrying capacity’ AAM Sustainability icon  Information and communication technology capability icon Civics and citizenship icon
	+ verify (by substitution) that a solution to the differential equation is , for an arbitrary constant , and a fixed quantity, and that the solution is in the case when
 | Introducing the modified exponential model* Build on the simple exponential models to develop the modified exponential models for growth and decay, relating it to the vertical translation of curves.
* Staff may like to use the following activities:
	+ Modelling modified exponential growth using dice. **Resource**: modelling-modified-exponential-growth-using-dice.DOCX and Sample-Data.XLSX
	+ Modelling modified exponential decay using dice. **Resource**: modelling-modified-exponential-decay-using-dice.DOCX
	+ Modelling the temperature of a cup of boiling water placed in a cooler environment. **Resource**: modelling-the-temperature-of-a-cup-of-boiling-water.DOCX

Verifying the modified exponential model* Students need to verify the modified exponential models algebraically

and rearranging we get: Note: is an arbitrary constant (difference between initial quantity and fixed)  is a fixed quantity The solution is N equals P in the case when A equals 0 |       |       |
| Representing the modified exponential model as a curve(2 lessons) | * establish the modified exponential model, for dealing with problems such as ‘Newton’s Law of Cooling’ or an ecosystem with a natural ‘carrying capacity’ AAM Sustainability icon  Information and communication technology capability icon Civics and citizenship icon
	+ sketch the curve for positive and negative values of
	+ note that whenever the quantity tends to the limit as irrespective of the initial conditions
	+ recognise that this model states that the rate of change of a quantity varies directly with the difference in the size of the quantity and a fixed quantity at any instant
 | Sketching the curve for the modified exponential model for positive and negative values of k* Geogebra Applet to investigate the curve for the modified exponential growth and decay. **Resource**: geogebra-applet–graphing-modified-exponential-growth-and-decay.GGB
	+ Examine a range of graphs as a class (or individually) to consider the following (some sample responses provided):
	+ What effect does the value of k have on the curve?

When k is positive the graph represent exponential growth.When k is negative the graph represent exponential decay.A larger (absolute) k value is graphically shown through a faster rate of growth or decay (steeper graph).* + How is P and/or A represented on the graph?

Consider the y-intercept and that whenever , the quantity N tends to the limit P as , irrespective of the initial conditions.P is the natural carrying capacity of a population, or when we consider a cooling object, the ambient (room) temperature that it is sitting in, refer to the results of Resource 14: Modelling the temperature of a cup of boiling water placed in a cooler environment* + When sketching the graph of what information should be included?
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| Applying modified exponential models to real life situations(1 lesson) | * solve problems involving situations that can be modelled using the exponential model or the modified exponential model and sketch graphs appropriate to such problems AAM Critical and creative thinking icon
 | Applying the modified exponential models to real life situations* Staff may like to run the following activities:
	+ Teacher guide to finding another way to determine k. **Resource**: another-way-to-determine-k.DOCX
	+ A matching activity to match conditions to modified exponential models. **Resource**: matching-activity-for-modified-growth-and-decay-model.DOCX
	+ A matching activity for Newton’s Law of Cooling. **Resource**: matching-activity-for-Newtons-Law-of-Cooling.DOCX
	+ A matching activity for heating to an ambient temperature. **Resource**: matching-activity-for-heating-to-the-ambient-temperature.DOCX
	+ The following activities provide students opportunities to apply modified models of exponential growth and decay, including
	+ Newton’s Law of Cooling
	+ Time of Death
	+ Problem Solving Activities

**Resource**: applications-of-modified-exponential-growth-and-decay.DOCXDuring activities students need to consider, is the simple growth or modified growth model better for a given scenario? |       |       |
| Solving problems using related rates(2 or 3 lessons) | **C1.3: Related rates of change*** solve problems involving related rates of change as instances of the chain rule (ACMSM129) AAM  Information and communication technology capability icon
* develop models of contexts where a rate of change of a function can be expressed as a rate of change of a composition of two functions, and to which the chain rule can be applied Critical and creative thinking icon
 | **Establishing how rates relate to each other*** Staff need to establish the chain rule which shows how a rate of change can be determined from two (or more) linking or related rates
* Staff need to build on the idea of a derivate as an infinitesimal fraction and establish (the reciprocal of )

**Solving Problems using related rates*** **Note:** During this topic students will need to identify a relationship linking two quantities, generally a relationship linking volume or area to height or length.
* Note: Students will need to differentiate the relationship to determine one of the related rates and use it in conjunction with a rate of change given to develop a resulting rate of change using the chain rule.
* Staff may like to run the following activities
	+ [Balloon change activity](http://mathiscool.org/confactivities/balloon.pdf), by mathsiscool.org which could be delivered at the start of the topic so that students can experience the changing quantities and the relationships between them.
	+ [4 Steps to Solving Related Rates Problems](https://www.matheno.com/blog/4-steps-to-solve-related-rates-problem-part-1/) by Matheno.com with examples
	+ [applications of differentiation](https://www.amsi.org.au/ESA_Senior_Years/PDF/AppsDiff3c.pdf)  provided by the Australian Mathematical Sciences Institute
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Reflection and evaluation

Please include feedback about the engagement of the students and the difficulty of the content included in this section. You may also refer to the sequencing of the lessons and the placement of the topic within the scope and sequence. All ICT, literacy, numeracy and group activities should be recorded in the ‘comments, feedback, additional resources used’ section.