 A formula for prime numbers

A formula that gives prime numbers

The following statement claims that a prime number can be created from any natural number using the formula given below. A proof by induction is given to support the answer.

For every natural number, $n$, the numbers $n^{2}-n+41$ is prime.

Proof

Let us check these claims:

* Claim (1) is true, as $1^{2}-1+41=41$ is a prime number.
* Claim (2) is true, as $2^{2}-2+41=43$ is a prime number.
* Claim (3) is true, as $3^{2}-3+41=47$ is a prime number.
* Claim (4) is true, as $4^{2}-4+41=53$ is a prime number.
* Claim (5) is true, as $5^{2}-5+41=61$ is a prime number . . .

Continuing in this way, we can see that the number $n^{2}-n+41$ is prime, for every natural number $n.$

Answer the following questions:

1. Explain why this proof is not a correct proof by mathematical induction.

1. Use an excel spreadsheet to calculate$ n^{2}-n+41$, for $n=1,2,3,…50.$

1. Find a list of the first 300 prime number using Google.

1. Can you find a number in your spreadsheet which is not listed as a prime number?

What is that number?

1. Check the value of $n$ used in the formula to generate that number and explain why it doesn’t give a prime number?

1. Try to complete the proof by induction starting with $n=1$. Highlight or circle and comment on the steps that are difficult to complete in this proof.

1. Both questions 5 and 6 prove that the formula is not true for all natural numbers. Write a conclusion justifying the application of proof by induction in checking such statements.

Solutions

The following statement claims that a prime number can be created from any natural number using the formula given below. A proof by induction is given to support the answer.

For every natural number $n, $the number $n^{2}-n+41$ is prime.

Proof

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* Claim (5) is true, as $5^{2}-5+41=61$ is a prime number . . . .

Continuing in this way, we can see the number $n^{2}-n+41$ is prime, for every natural number $n.$

Answer the following questions:

Question 1

Explain why this proof is not a correct proof by mathematical induction.

Answer

The step for assuming the initial statement to be true for $n=k$ where $k$ is a natural number is missing.

The inductive step for $n=k+1$ is missing.

Conclusion is missing.

In particular, the rule working for $n=5$ is not caused by or proven based on the rule working for
$$n=4.$$

Question 2

Use an excel spreadsheet to calculate $n^{2}-n+41,$ for $n=1,2,3,…50.$

Answer

See prime-number-formula.XLSX

Question 3

Find a list of the first 300 prime numbers using google.

Answer

See prime-number-formula.XLSX

Question 4

Can you find a number in your spreadsheet which is not listed as a prime number? What is that number?

Answer

1681

Question 5

Check the value of n used in the formula to generate that number and explain why it doesn’t give a prime number.

Answer

$n=41$ was used to generate 1681.

If we substitute $n=41$ in the formula, it gives

$$41^{2}-41+41=41^{2}$$

Clearly $41^{2}$ is not a prime number.

Question 6

Try to complete the proof by induction starting with $n=1.$ Highlight or circle and comment on the steps that are difficult to compete in this proof.

Answer

Induction hypothesis $n^{2}-n+41$ gives prime numbers.

Step 1

Test if the result is true for $n=1$

LHS$=1^{2}-1+41…=41$

41 is a prime number. $∴True for n=1$

Step 2

Let $n = k$ be a value for which the result is true where k is a natural number.

For example: $k^{2}–k+41 = P $where P is a prime number. (Difficulty 1: What does this mean and how can we write it algebraically?)

Step 3

Test if the result will then be true for $n=k+1$

For example: To prove

$(k+1)^{2}-\left(k+1\right)+41=Q$ where $Q$ is a prime number too. (Difficulty 2: Again, what does this mean and how can it be expressed algebraically? Hence, what are we trying to prove?)

LHS$=k^{2}+2k+1-k-1+41$

$$=k^{2}+k+41$$

$$=k^{2}-k+41+2k$$

$=P+2k$ (Difficulty 3: How do we prove this is a prime number?)

Note – There are reasons to suggest that $P+2k$ will be even, and since P is odd (prime and greater than 2), the sum of an odd number (P) and an even number (2k) will always be odd. However being odd is not enough to conclusively prove that the number $P+2k$ is prime. Hence the proof by mathematical induction is incomplete and inconclusive.

Question 7

Both questions 5 and 6 prove that the formula is not true for all natural numbers. Write a conclusion justifying the application of proof by induction in checking such statements.

Answer

The formula is true for first 40 numbers so could have been mistakenly taken as true for all natural numbers. Our inability to prove the result by mathematical induction highlights the chance that this is not true for all natural numbers, and gives rise to our investigation.