 Year 11 mathematics advanced

| MA-T1 Trigonometry and measure of angles Paperclip icon | Unit duration |
| --- | --- |
| The topic Trigonometric Functions involves the study of periodic functions in geometric, algebraic, numerical and graphical representations. The study of trigonometric functions is important in developing students’ understanding of periodic behaviour, a property not possessed by any previously studied functions. Utilising this property, mathematical models have been developed that describe the behaviour of many naturally occurring periodic phenomena, such as vibrations or waves, as well as oscillatory behaviour found in pendulums, electric currents and radio signals. | 4-5 weeks |

| Subtopic focus | Outcomes |
| --- | --- |
| The principal focus of this subtopic is to solve problems involving triangles using trigonometry, and to understand and use angular measure expressed in radians and degrees. This has practical and analytical applications in areas including surveying, navigation, meteorology, architecture, construction and electronics. Students develop techniques to solve problems involving triangles, and then extend these ideas to include the exact ratios for angles, and also to the study of non-right-angled triangles. This introduces the need to define the trigonometric ratios for obtuse angles, which is followed by the establishment of trigonometric ratios of angles of any size. Radians are introduced as another measure in which angles of any size can be found. Radians are important for the study of the calculus of trigonometric functions in Year 12. Within this subtopic, schools have the opportunity to identify areas of Stage 5 content which may need to be reviewed to meet the needs of students. | A student:  uses algebraic and graphical techniques to solve, and where appropriate, compare alternative solutions to problems MA11-1  uses the concepts and techniques of trigonometry in the solution of equations and problems involving geometric shapes MA11-3  uses appropriate technology to investigate, organise, model and interpret information in a range of contexts MA11-8  provides reasoning to support conclusions which are appropriate to the context MA11-9 |

| Prerequisite knowledge | Assessment strategies |
| --- | --- |
| Students should have studied Stage 5.2 right-angled triangles, Stage 5.3 trigonometry and Pythagoras’ theorem, algebraic techniques, equations and properties of geometrical figures. | Student self-assessment using [KHAN Academy](https://www.khanacademy.org/), [Maths Warehouse](http://www.mathwarehouse.com/trigonometry/radians/convert-degee-to-radians.php), [Quizlet](https://quizlet.com/64782126/6-trig-functions-of-0-90-180-270-360-flash-cards/),  Pre-tests and quizzes designed by teacher to revise previous concepts  **Assessment:** How are outdoor concert spaces designed? |

All outcomes referred to in this unit come from [Mathematics Advanced](http://educationstandards.nsw.edu.au/wps/portal/nesa/11-12/stage-6-learning-areas/stage-6-mathematics/mathematics-advanced-2017) Syllabus  
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Glossary of terms

| Term | Description |
| --- | --- |
| ambiguous case  | In trigonometry, the ambiguous case refers to using the sine rule to calculate the size of an angle in a triangle where there are two possibilities for the angle, one obtuse and one acute, leading to two possible triangles. |
| radian | One radian is the angle subtended at the centre of a circle by an arc of 1 unit in length and the radius of 1 unit in length. |
| periodicity | The quality of recurring at regular intervals. |

| Background knowledge |
| --- |
| Almost all early civilisations used the shadow cast by a vertically positioned stick to observe the motion of the sun and tell time. This instrument is now called a Gnomon, the Greek name of an L-shaped instrument used to draw a right angle. Tables of the numerical sequences of Gnomon shadow lengths and correlating times of day are often viewed as ancestors to the tangent and cotangent.  The foundations of trigonometry were laid as early Babylonian, Greek, Hellenistic, Indian, and Arabic mathematicians investigated astronomical problems using numerical and geometric techniques, specifically the geometry of the circle. Hipparchus of Nicaea (c.190–120BC) is often referred to as the ‘father of trigonometry’ because the earliest evidence of trigonometric tables relating to chord lengths was attributed to him. As an astronomer, Hipparchus, focused mainly on spherical triangles, such as those formed between three stars.  Trigonometry was established as a distinct branch of mathematics during the 12th and 13th centuries. However it was not until the 16th century that trigonometry moved from being predominantly geometric to an algebraic-analytic discipline.  Students may be interested in researching the history of the terms used in trigonometry, and this can assist in understanding certain concepts. For example, terms such as sine and cosine evolved within the context of astronomy and spherical geometry, whereas tangent and cotangent from shadows and the Gnomon. The first trigonometric ratios to be calculated were related to the sine ratio. Early expressions for cosine included ‘sinus complementi’ and ‘cosinus’. As these expressions indicate, the cosine of an angle is the sine of the complement of that angle, so that once a set of values has been established for the sine ratio, these can be used for the cosine ratio as well. For example, the cosine of 30° is the sine of the complement of 30° and is therefore the same value as the sine of 60°. |

Lesson sequence

| Lesson sequence | Content  Students learn to: | Suggested teaching strategies and resources | Date and initial | Comments, feedback, additional resources used |
| --- | --- | --- | --- | --- |
| Review of Stage 5 right angled trigonometry  (1 – 2 lessons) | * use the sine, cosine and tangent ratios to solve problems involving right-angled triangles where angles are measured in degrees, or degrees and minutes Paperclip icon | **Stage 5 review**  Teacher to pre-quiz students on their knowledge of Pythagoras’ theorem, converting and rounding angle measures and using the sine, cosine and tangent ratios to find sides and angles to determine the amount of review required from Stage 5. Students can self-assess their prior knowledge using Khan Academy: [Pythagoras’ Theorem](https://www.khanacademy.org/math/basic-geo/basic-geometry-pythagorean-theorem/geo-pythagorean-theorem/e/pythagorean_theorem_1), [Finding an unknown side](https://www.khanacademy.org/math/geometry/hs-geo-trig/hs-geo-solve-for-a-side/e/trigonometry_2) and [Finding an unknown angle](https://www.khanacademy.org/math/geometry/hs-geo-trig/hs-geo-solve-for-an-angle/e/solve-for-an-angle-in-a-right-triangle)  Cover common misconceptions with angle measures, for example, some students incorrectly think that . Revise how to round angles to the nearest degree and nearest minute.  Students to apply these skills to solving problems involving right-angled triangles.  **Complementary angle relationships**  The following complementary angle relationships should be discovered and explored:  , where is an acute angle  , where is an acute angle  Students may investigate by constructing a right-angled triangle (for example – the Pythagorean triangle with sides equal to 3, 4 and 5). In pairs have one student find the sine and cosine ratios for one of the acute angles, and the other is to find the sine and cosine ratios for the other acute angles. Students to compare answers, observe the similarities and determine the relationship between the two angles. Ready-made worksheets are available from [C Palms](http://www.cpalms.org/Public/PreviewResourceLesson/Preview/130872) website. |  |  |
| Sine rule, cosine rule and area of a triangle formula  (2 – 3 lessons) | **T1.1: Trigonometry**   * establish and use the sine rule, cosine rule and the area of a triangle formula for solving problems where angles are measured in degrees, or degrees and minutes AAM Paperclip icon  Information and communication technology capability icon * find angles and sides involving the ambiguous case of the sine rule * use technology and/or geometric construction to investigate the ambiguous case of the sine rule when finding an angle, and the condition for it to arise Critical and creative thinking icon  Information and communication technology capability icon | **Sine rule**  Demonstrate proof of sine rule, however, students will not be expected to reproduce the proof. The [mathematics online clip](https://www.youtube.com/watch?v=3jBMymLI8ls) (duration 6:50) clearly shows the proof of the cosine rule.  Teacher to clarify with students that the sine rule is used for non-right angled triangles. Students should be familiar with various forms of the sine rule and how to change the subject of the formula.  Students should be able to use the various forms as follows:  , used to find a missing side, given its opposite angle and one other pair of a side and its opposite angle or  used to find a missing angle, given its opposite side and one other pair of a side and its opposite angle  Demonstrate the ambiguous case of the sine rule geometrically and algebraically to show that two possible triangles may exist when finding an angle. Teachers can use the pre-made [Geogebra resource](https://www.geogebra.org/m/MfFKUj2q) to demonstrate this geometrically or use [SoftSchools](http://www.softschools.com/math/calculus/the_ambiguous_case_of_the_law_of_sines/) to demonstrate both geometrically and algebraically.  Teacher should demonstrate a variety of examples on finding the missing angle and side using the sine rule.  **Cosine rule**  Demonstrate proof of cosine rule, however, students will not be expected to reproduce the proof. Emphasise that Pythagoras’ theorem is a special case of the cosine rule. The [mathematics online clip](https://www.youtube.com/watch?v=3jBMymLI8ls) (duration 6:50)clearly shows the proof of the cosine rule.  Teacher to clarify with students that the cosine rule is used for non-right angled triangles. Students should be familiar with various forms of the cosine rule and how to change the subject of the formula.  Students should be able to use the various forms as follows:  , used to find a missing side, given the other two sides and their included angle or  , used to find a missing angle, given the three side lengths of the triangle  Teacher should demonstrate a variety of examples on finding the missing angle and side using the cosine rule, including more complex questions.  NESA exemplar question  Find the value of in the following diagram.  The diagram is of a triangle with sides 13, (x+4) and (x-4) units. The angle opposite the side of 13 is 60 degrees.  **Sine and cosine rule**  Students to work through questions where there is a mixture of problems that use both the sine and cosine rules, including real-world problems. At times students have trouble determining which rule to use.  **NESA exemplar question**  Determine the possible dimensions for triangle given  cm, and .  **Area of a triangle**  Demonstrate proof of the area of a triangle rule , however, students will not be expected to reproduce the proof. This can be done by showing the [fuse school clip](https://www.youtube.com/watch?v=JqIazlNOrFQ) (duration 1:20). Teachers should emphasise that students should still use formula when they know the base length and the perpendicular height.  Emphasise that the sides used in the formula must be the arms of the angle used. Provide students with questions that give all three sides and one angle or two sides and two angles so that they must first determine which sides or angles to use.  NESA exemplar question  In the diagram, is a sector of the circle with centre and radius 6 cm, where . Determine the value of the area of the triangle .  The diagram illustrates a sector of 30 degrees with a radius of 6 cm. the segment of the sector is shaded. |  |  |
| Solving practical problems in trigonometry  (1 – 2 lessons) | * solve practical problems involving Pythagoras’ theorem and the trigonometry of triangles, which may involve the ambiguous case, including finding and using angles of elevation and depression and the use of true bearings and compass bearings in navigation AAM Paperclip icon Critical and creative thinking icon | **Angles of elevation and depression**  Review angles of elevation and depression as required. Students can self-assess using [Khan Academy](https://www.khanacademy.org/math/geometry/hs-geo-trig/hs-geo-modeling-with-right-triangles/a/angles-of-elevation-and-depression).  Teacher should ensure that students make the connection that angles of elevation and depression are equal as they are alternate angles.  Students should be exposed to questions where the diagram is provided, as well as questions where they need to construct their own diagram.  Emphasise that the angle of depression is on the outside of the triangle, this is a common misconception of students when drawing their own diagrams.  **Bearings**  Review true and compass bearings as required. Both true and compass bearings are to be covered.  NESA examples of how compass bearings can be expressed:  one of the 16 points of a mariners compass, for example – SSW  the number of degrees east or west of the north-south line, for example – N30°E, S80°W  common descriptions, for example – ‘due East’, ‘South-West’.  The common descriptions and 16 points of a mariners compass can be shown with a diagram such as:  16 point mariners compass  For other compass bearings the teacher should explain that angles are always measured from north or south, whichever is closer. Then the number or degrees travelled east/west gives the bearing. For example – . N30°E. True bearings are the three-figure bearings, for example – 025°, 240°. Teacher should explain that true bearings are always measured clockwise from the north line.  Teacher should provide student with questions that required them to draw a bearing or read a bearing off a given diagram. Some examples of this can be found on the [MathsTeacher Site](http://mathsteacher.com.au/year10/ch15_trigonometry/11_directions/23dir.htm)  Students should also be exposed to examples that involve both right-angled and non-right angled trigonometry, as well as the ambiguous case for the sine rule.  NESA exemplar question  The diagram illustrates the information that is presented in the wording of the question.  Chris leaves island in a boat and sails 142 km on a bearing of to island . Chris then sails on a bearing of for 220 km to island , as shown in the diagram below.  Show that the distance from island to island is approximately 210 km.  Chris wants to sail from island directly to island . On what bearing should Chris sail? Give your answer to the nearest degree.  **Note –** problems should be provided that are given diagrammatically as well as problems in written form. |  |  |
| Two and three dimension trigonometry problems  (1 – 2 lessons) | * solve problems involving the use of trigonometry in two and three dimensions AAM Paperclip icon * interpret information about a two or three-dimensional context given in diagrammatic or written form and construct diagrams where required | **Problems in two and three dimensions**  Teacher to provide students with a problem in three dimensions, such as drawing the diagonal of a rectangular prism, or two people at different locations looking up at a building.  It can be helpful to have props as students can struggle to visualise 3D trigonometry when it is drawn as a two dimensional diagram, it is important that the teacher can bring the problem to life.  Teacher to explain that the key to answering questions in 3D trigonometry is thinking about separate triangles and parts of them in 2D, as well as drawing all the triangles that can be seen to help visualise what to do.  Using the two people looking up at a building examples, see if students can visualise how many two dimensional triangles (right and non-right) are in the problem. Ensure that the students can visualise the two triangles formed by the angles of elevation from each person, as well as the triangle made on the ground between the two people and the base of the building. There are other triangles, although these are the three mainly used in this style of question.  Once students can visualise the 3D situations and can pull apart 3D trigonometry to see 2D triangles, the teacher should work through examples starting with basic questions such as the diagonal of a prism drawn where the diagram is provided.  NESA exemplar questions  The Eiffel Tower is located in Paris, a city built on a flat floodplain. Three tourists , and are observing the Eiffel Tower from the ground. is due north of the tower, is due east of the tower, and is on the line-of-sight from to and between them. The angles of elevation to the top of the Eiffel Tower from , and are , and , respectively. Determine the bearing of from the Eiffel Tower.  A person walks 2000 metres due north along a road from point to point . The point is due east of a mountain , where is the top of the mountain. The point is directly below point and is on the same horizontal plane as the road. The height of the mountain above point is metres. From point , the angle of elevation to the top of the mountain is . From point , the angle of elevation to the top of the mountain is . Determine the height of the mountain.  AB is 2000 m. A is due east of a mountain OM, where M is the top of the mountain. The point O is directly below point M and is on the same horizontal plane as AB. The height of M above O is h metres.  From point A, the angle of elevation to M is 15°.  From point B, the angle of elevation to M is 13°.  Students could either be given a diagram for a three-dimensional problem or construct a diagram from the information provided. Teachers should ensure that students are exposed to both types of questions, and it should be emphasised that a diagram must be drawn by the student when it hasn’t been provided. |  |  |

| Lesson Sequence | Content  Students learn to: | Suggested teaching strategies and resources | Date and Initial | Comments, Feedback, Additional Resources Used |
| --- | --- | --- | --- | --- |
| Introduction to radians  (1 lesson) | **MA-T1.2: Radians**   * define and use radian measure and understand its relationship with degree measure (ACMMM032) Literacy icon * convert between the two measures, using the fact that radians | **Introduction to radians**  Students have only been working with angles that are measured in degrees in trigonometry. They will now be introduced to radians.  For a given angle, the ratio of the length of the arc it subtends to the radius of the arc provides the natural measure of angles, and is called circular measure (or radian measure). Using circular measure, sine and cosine are now defined as functions of a real variable: for each real number , is defined as the sine of an angle whose circular measure is , is defined as the cosine of this angle. (NESA)  It should be noted that a ‘radian’ is not a unit of measure. It has no dimensions. However, the term is often used to indicate that an angle is given in circular measure and not in degrees. For example, means the sine of an angle whose circular measure is , and this may be read as ‘ radians’ to distinguish it from , the sine of an angle of 3 degrees. (NESA)  One radian is the angle subtended at the centre of a circle by an arc of 1 unit in length and the radius of 1 unit in length. The [WikiMediaCommons GIF](https://commons.wikimedia.org/wiki/File:Circle_radians.gif#/media/File:Circle_radians.gif) shows this visually.  The relationship between degrees and radians should be investigated and deduced by the students as is equivalent to radians. To convert from degrees to radians multiply by   pi over 180  and to convert from radians to degrees multiply by .  Students can self-assess their competency for converting angles between degrees and radians using [MathWarehouse](http://www.mathwarehouse.com/trigonometry/radians/convert-degee-to-radians.php). |  |  |
| Exact values  (1 lesson) | * define and use radian measure and understand its relationship with degree measure (ACMMM032) * recognise and use the exact values of , and in both degrees and radians for integer multiples of and (ACMMM035) | **Exact values**  Exact value triangles were introduced in the Stage 5.3 course, with all angles measured in degrees. Practice should be given so that exact equivalents are known for common angle sizes and so that accuracy is developed in approximating sizes given in one measure by sizes in another. The ratios for and radians should be known as exact values.  The right angled isosceles and equilateral triangles below should be developed with students and used as the exact value triangles. These triangles need to be shown using both degrees and radians.  See the source image  **NESA exemplar questions**  Find the exact values of:  Convert radians to degrees.  Find the exact value of . |  |  |
| Sketching trigonometric functions in degrees and radians (1 – 2 lessons) | * understand the unit circle definition of and and periodicity using degrees (ACMMM029) * sketch the trigonometric functions in degrees for * recognise the graphs of , and and sketch on extended domains in degrees and radians (ACMMM036) | Sketching trigonometric functions  Students should establish the trigonometric ratios for , and from the definitions. Students can use the calculator to confirm these results. They will make connections to the unit circles at a later stage.  Students can test their knowledge of these values using [Quizlet](https://quizlet.com/64782126/6-trig-functions-of-0-90-180-270-360-flash-cards/).  The graphs should be drawn showing the ratios , and as functions of the angular measure in degrees. The graphs should also be explored using radian measure. Teachers should discuss the benefits of using radian measure over degrees when graphing the trigonometric functions in relation to the scales of the axes.  The functions are defined for all real. Their graphs should be drawn using computer software, graphing calculators, or by-hand methods. The function may then be defined in terms of and . That is; the domain of definition found, and the graph drawn.  A pre-made [DESMOS](https://www.desmos.com/calculator/q02tsdydbv) template can be used to graph each trigonometric function, showing each quadrant and the magnitude of each trigonometric function in radian measure.  Students to use their knowledge and the trigonometric graphs to create the summary below from [Maths Links – Steven Choi](https://mathslinks.net/):  , , , use the graph of  , , , use the graph of  , use the identity  , use the identity |  |  |
| Introduction to the unit circle and angles of any magnitude  (1 – 2 lessons) | * understand the unit circle definition of , and and periodicity using degrees (ACMMM029) * understand the unit circle definition of , and and periodicity using radians (ACMMM034) | Unit circle  To explore the concept of angles of any magnitude, students need to understand why the -coordinate is cosine and the -coordinate is sine. This should be illustrated with reference to the unit circle and the Pythagorean relationship to establish .  , and should be defined for any angle with relation to the unit circle. This can clearly be shown using [Interactive Unit Circle](https://www.mathsisfun.com/geometry/unit-circle.html), and the angles that should be defined include .  Using the and coordinates, students can determine the sign of sine, cosine and tangent for angles greater than 90 in both radians and degrees. Students can connect the unit circle to the trigonometric graphs using the pre-made [Geogebra Unit Circle for Sine and Cosine](https://www.geogebra.org/m/Z26WBQgM), pre-made [Geogebra Trigonometric Unit Circle with Exact Values](https://www.geogebra.org/m/tCkm2E23) and pre-made [Geogebra Unit Circle and Tangent Graph](https://www.geogebra.org/m/cf6KYJeb).  Students are to compare the trigonometric values in each quadrant and state where each trigonometric quadrant is positive. The term ‘related angle’ should be introduced. For example, compare the value of and using the unit circle. Students may recall aids related to which quadrants of the unit circle contain positive results for , and could be used such as CAST or mnemonics like All Stations To Central (ASTC).  Trigonometric ratios for angles which can be written in the form  , and , where should be obtained. Students can investigate this by working out the acute angle from the x axis on the unit circle, students can use [Geogebra pre made interactive](https://www.geogebra.org/m/j2mf8aKJ) to assist. The following supplementary angle relationships should be explored.  , where is an acute angle  , where is an acute angle  where is an acute angle.  Explore and use the results for reflex angle relationships where :  ,  ,  ,  ,  Students should note that angles greater than 360° or occur when making more than one complete revolution. Students can investigate this idea by looking at angles greater than 360 on the unit circle and comparing them with their equivalent acute angle. The [pre-made Geogebra interactive](https://www.geogebra.org/m/rAmpPfTP) could be used to assist in the demonstration/discovery. Related angles should also be revisited here, for example, to explore that 30 will be equivalent to  NESA exemplar question  Consider a circular clock-face of radius 1 unit centred on the origin. The coordinates of the ‘twelve o’clock’ position are (0,1) and the coordinates of the ‘one o’clock’ position are .  What are the coordinates of the other hour positions?  Identify the value of in each case by writing each pair of coordinates in the form .  Note that this can be used to reinforce the exact values of the trigonometric ratios for angles such as 30°, 60°, 90°, 120°,150°, … |  |  |
| Solving problems involving angles of any magnitude  (1 lesson) | * solve problems involving trigonometric ratios of angles of any magnitude in both degrees and radians | Once familiarity with the trigonometric ratios of angles of any magnitude is attained, some practice in solving simple equations, of the type likely to occur in later applications, should be discussed. Solutions should be given in exact form where possible.  Students should revisit the solution of a problem involving the ambiguous case of the sine rule, using a pre-made [Geogebra Interactive](https://www.geogebra.org/m/DAQZqfQ9), the unit circle and angles of any magnitude to appreciate that this is an example of the use of the related angle , where and or for .  **NESA exemplar questions**  Solve for x or include:    .  for  for |  |  |
| Arc length and area of a sector formula  (1 – 2 lessons) | * derive the formula for arc length, and for the area of a sector of a circle, Literacy icon * solve problems involving sector areas, arc lengths and combinations of either areas or lengths | Deriving the arc length and area of a sector  The proof that an angle of has a circular measure of using a circle has previously been explored. Students will use this relationship to derive the formula , for the length of an arc subtending an angle at the centre of a circle of radius . The formula for the area of the corresponding sector should also be derived. These could be explored using [Proof Wiki Arc Length of Sector](https://www.geogebra.org/m/DAQZqfQ9) and [Proof Wiki Area of Sector](https://proofwiki.org/wiki/Area_of_Sector), although encourage students to develop the formulas on their own prior to showing the websites. It is important for students to ensure that their angle is measured in radians when using these formulas.  Students may like to extend their knowledge of the area of a triangle and the area of a sector to derive the formula for the area of a segment as  Students should also be reminded of both the sine and cosine rule for finding sides and angles when completing questions.  NESA exemplar questions  Find the perimeter and the area of the segment cut off by a chord of length 8 cm in a circle centre and radius 6 cm. Give your answers correct to 3 significant figures.  If a chord of a circle which subtends an angle of at the centre of the circle cuts off a segment equal in area to of the area of the whole circle.  Show that .  Verify that radians, correct to 2 decimal places. |  |  |

Reflection and evaluation

Please include feedback about the engagement of the students and the difficulty of the content included in this section. You may also refer to the sequencing of the lessons and the placement of the topic within the scope and sequence. All ICT, literacy, numeracy and group activities should be recorded in the ‘comments, feedback, additional resources used’ section.