 Year 11 mathematics advanced

| MA-E1 Logarithms and exponentials | Unit duration |
| --- | --- |
| The topic Exponential and Logarithmic Functions introduces exponential and logarithmic functions and develops their properties, including the manipulation of expressions involving them. The exponential function is introduced by considering graphs of the derivative of exponential functions. A knowledge of exponential and logarithmic functions enables an understanding of practical applications, such as exponential growth and decay, as well as applications within the Calculus topic. The study of exponential and logarithmic functions is important in developing students’ ability to solve practical problems involving rates of change in contexts such as population growth and compound interest | 3 weeks |

| Subtopic focus | Outcomes |
| --- | --- |
| The principal focus of this subtopic is for students to learn about Euler’s number 𝑒, become fluent in manipulating logarithms and exponentials and to use their knowledge, skills and understanding to solve problems relating to exponentials and logarithms. Students develop an understanding of numbering systems, their representations and connections to observable phenomena such as exponential growth and decay. The exponential and logarithmic functions and are important non-linear functions in Mathematics, and have many applications in industry, finance and science. They are also fundamental functions in the study of more advanced Mathematics. Within this subtopic, schools have the opportunity to identify areas of Stage 5 content which may need to be reviewed to meet the needs of students. | A student:* manipulates and solves expressions using the logarithmic and index laws, and uses logarithms and exponential functions to solve practical problems MA11-6
* uses appropriate technology to investigate, organise, model and interpret information in a range of contexts MA11-8
* provides reasoning to support conclusions which are appropriate to the context MA11-9
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| Prerequisite knowledge | Assessment strategies |
| --- | --- |
| Students should have studied Stage 5.3 index laws and non-linear functions, as well as MA-C1 Introduction to differentiation. | Students could use graphing software to investigate the transformations of exponential and logarithmic functions and explain the effect of varying each term in the equations using reasoning and communication skills.  |

All outcomes referred to in this unit come from [Mathematics Advanced](http://educationstandards.nsw.edu.au/wps/portal/nesa/11-12/stage-6-learning-areas/stage-6-mathematics/mathematics-advanced-2017) Syllabus
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Glossary of terms

| Term | Description |
| --- | --- |
| Euler’s number | The unique number, 2.71828182845, for which .  |
| logarithm | An index, exponent or power |

| Background knowledge |
| --- |
| John Napier (1550–1617), the Scottish mathematician, is often referred to as the inventor of logarithms. Napier was particularly famous not only for logarithms but also his various devices, such as ‘Napier’s bones’ which assisted in computations in the era before calculators. In 1614 he published his book titled *Mirifici logarithmorum canonis descriptio* which translates to ‘A Description of the Wonderful Table of Logarithms.’It is interesting to note that within years of one another at least one other scholar, Joost Burgi (1552–1632), independently of Napier, also created systems involving logarithmic relations and produced tables for their use. Exploration of this intricate history of logarithms may be of interest to students as may be study of the connection to arithmetic and geometric sequences. Logarithms and Napier’s work facilitated countless advances in areas such as engineering and science by making complex calculations possible before the advent of the electronic calculator. Calculations requiring tedious multiplications and divisions were carried out using logarithms and logarithmic tables. For example, if a person wanted to multiply two large numbers together they would convert each number to a logarithm by looking them up in logarithmic tables. These two logarithms were then added and the tables were used once again to convert this sum to the required product. This process was often much faster and accurate than performing the multiplication by hand and is the principle upon which *slide rules* are made. |

| Lesson sequence | ContentStudents learn to: | Suggested teaching strategies and resources  | Date and initial | Comments, feedback, additional resources used |
| --- | --- | --- | --- | --- |
| Review index laws and introduce logarithms and exponentials as inverse operations(1 – 2 lessons) | **E1.1 Introducing logarithms*** define logarithms as indices*:*  is equivalent to *,* and explain why this definition only makes sense when
* recognise and sketch the graphs of *,* where is a constant, andPaperclip icon
* recognise and use the inverse relationship between logarithms and exponentials
	+ understand and use the fact that for all real , and for all
 | **Assumed knowledge**Review of index laws as required. This content has already been captured in MA-F1. It may be necessary to recap in order to make the relationships between index and logarithmic laws more explicit.Investigating exponential functions* Investigate graphs of exponential functions using the activity “Can folding paper get you to the moon?” **Resource:** can-folding-paper-get-you-to-the-moon.DOCX
* Students can use DESMOS to play the game [polygraph exponentials](https://teacher.desmos.com/polygraph/custom/56c3947ce3a0912c0a942de0) to learn about the features of exponential graphs

Introduction to logarithmic functions* Investigate the inverse of exponentials using mira mirrors or any means to develop understanding of the need for logarithms.
* Demonstrate the relationship between exponential and logarithmic functions using the Geogebra template [Exponential and Logarithmic Functions](https://www.geogebra.org/m/NwaefQKz)
* Numerical and algebraic introduction to logarithms as the inverse of an exponential expression. For example, if then
* Converting between index form and log form [jigsaw](https://www.tes.com/en-au/teaching-resource/simplifying-expressions-involving-e-x-and-lnx-6148638)

NESA exemplar questions* Calculate to three decimal places.
* Convert the following to exponential form: .
* Convert the following to logarithmic form .

**Resource**: ma-e1-nesa-exemplar-questions-solutions.DOCX |       |       |
| Logarithmic laws and change of base(1 – 2 lessons) | **E1.2 Logarithmic laws and applications*** derive the logarithmic laws from the index laws and use the algebraic properties of logarithms to simplify and evaluate logarithmic expressions

, , , * consider different number bases and prove and use the change of base law AAM Aboriginal and Torres Strait Islander histories and cultures icon Difference and diversity icon

**E1.1 Introducing logarithms*** recognise and use the inverse relationship between logarithms and exponentials
	+ understand and use the fact that for all real , and for all
 | Deriving and using the log laws* Scaffolded proofs of the logarithmic laws, including the change of base law. Students are not expected to reproduce the derivations and are not expected to memorise the change of base law.
* Students should be encouraged to use mental techniques to evaluate simple logarithms. For example, .
* Investigate

NESA exemplar questions* Evaluate the following (without a calculator): , , and
* Estimate the value of by considering powers of
* Solve
* Solve the following: , and

**Resource**: ma-e1-nesa-exemplar-questions-solutions.DOCX |       |       |
| Real life applications of logarithmic scales(1 lessons) | * interpret and use logarithmic scales, for example decibels in acoustics, different seismic scales for earthquake magnitude, octaves in music or pH in chemistry (ACMMM154) AAM
 | Applying logarithmic scales in real life situations* Students to investigate real life applications of logarithmic scales, which could include but are not limited to:
	+ [Google’s PageRank](https://www.link-assistant.com/news/page-rank-2018.html)
	+ [Bill Nye the Science Guy explains Richter Scale](https://www.youtube.com/watch?v=1qbg7orb1lc) (duration 2:23)
	+ [What is a decibel and how is it measured?](https://science.howstuffworks.com/question124.htm)
* Students could complete a reflective exit slip “Which application did you find most interesting and why?”
 |       |       |
| Practical applications of problems involving logarithms(1 – 2 lessons) | * solve algebraic, graphical and numerical problems involving logarithms in a variety of practical and abstract contexts, including applications from financial, scientific, medical and industrial contexts AAM Critical and creative thinking icon

**E1.4 Graphs and applications of exponential and logarithmic functions*** solve equations involving indices using logarithms (ACMMM155)
 | Solving problems involving logarithms* Logarithms are used in obstetrics. When a woman becomes pregnant, she produces a hormone known as human chorionic gonadotropin. Since the levels of this hormone increase exponentially, and at different rates with each woman, logarithms can be used to determine when pregnancy occurred and to predict fetus growth.
* Students could review [How to calculate ph](https://www.thoughtco.com/how-to-calculate-ph-quick-review-606089) for those studying Science subjects
* Students could investigate the elimination of a drug from the body which follows exponential decay
* Students could research and consider [half life decay](https://www.coolmath.com/algebra/17-exponentials-logarithms/13-radioactive-decay-decibel-levels-01)
* Students could consider [Benford’s Law](https://www.youtube.com/watch?v=XXjlR2OK1kM) (duration 9:13)

**NESA exemplar questions*** On the Richter scale, the magnitude of an earthquake of intensity is given by the formula , where is a reference intensity used for comparisons.
	+ Find for an earthquake that is 4.3 million times more intense than the reference intensity.
	+ An earthquake measured 8.5 on the Richter scale. How many times more intense is this than the reference intensity?
* On the decibel scale, the loudness of a sound of intensity is given by , where is a reference intensity used for comparisons.
	+ A sound that causes pain in humans is about times more intense than . Find for a sound of this intensity.
	+ How many times more intense is the sound of a heated argument (about 67 decibels) than the sound of a quiet room (about 31 decibels)?
* The pH value of a solution is given by the formula , where is the concentration of hydrogen ions in moles per litre.
	+ Find values for each of the following: blood , beer .
	+ Find the concentration of hydrogen ions in moles per litre for the following: eggs (), water ().
* Solve the following: , , ,

**Resource**: ma-e1-nesa-exemplar-questions-solutions.DOCX |       |       |
| Investigation of Euler’s number and natural logarithms(2 lessons) | **E1.3 The exponential function and natural logarithms*** establish and use the formula (ACMMM100)
	+ using technology, sketch and explore the gradient function of exponential functions and determine that there is a unique number 2.71828182845, for which where is called Euler’s number  Information and communication technology capability icon
* apply the differentiation rules to functions involving the exponential function, , where and are constants
 | **Investigating natural logarithms*** Watch [e (Euler's Number) - Numberphile](https://www.youtube.com/watch?v=AuA2EAgAegE) (duration 10:41)
* Use the Geogebra worksheet [Derivatives of Exponential Functions](https://www.geogebra.org/m/Z48fCWAq) to explore the derivative of exponential functions and discover Euler’s number
* Use differentiation rules to differentiate

**NESA exemplar questions*** Find the gradient of the tangent to at: (i) (ii) (iii) . Write a statement in words linking the rate of change of and the value of at each point on the curve.
* If , simplify .
* Solve the equation: .
* Differentiate and hence find the gradient of the function when .

**Resource**: ma-e1-nesa-exemplar-questions-solutions.DOCX |       |       |
| Practical applications of problems involving natural logarithms(1 – 2 lessons) | * work with natural logarithms in a variety of practical and abstract contexts AAM
	+ define the natural logarithm from the exponential function (ACMMM159)
	+ recognise and use the inverse relationship of the functions and (ACMMM160)
	+ use the natural logarithm and the relationships where and for all real in both algebraic and practical contexts
	+ use the logarithmic laws to simplify and evaluate natural logarithmic expressions and solve equations
 | * The natural logarithmic function can be defined as the inverse of the exponential function. Students could use mira mirrors or graphing software to reflect the exponential curve in the line y=x to establish the results and
* Students should compare the graph of a natural logarithm with logarithms in other bases and notice the difference in the shape of the graphs
* The natural logarithmic function is commonly abbreviated by using or .
* [Demystifying the natural logarithm](https://betterexplained.com/articles/demystifying-the-natural-logarithm-ln/) contains a practical explanation of the usefulness of natural logarithms for describing the time it takes to reach a required level of growth.
* Consider the formula for calculating ‘Time of Death’ **Resource:** time-of-death.DOCX

Suggested applications and exemplar questions* Find the gradient of the tangent to at:
* Write a statement in words linking the rate of change of and the value of at each point on the curve.
* If , simplify .
* Solve the equation: .
* Differentiate and hence find the gradient of the function when .
 |  |  |
| Investigation of the transformations of logarithmic and exponential functions(1 lesson)  | **E1.4 Graphs and applications of exponential and logarithmic functions*** graph an exponential function of the form for and its transformations and where , and are constants Paperclip icon  Information and communication technology capability icon
* graph a logarithmic function  for and its transformations , using technology or otherwise, where and are constants  Information and communication technology capability icon
	+ recognise that the graphs of and are reflections in the line
 | Investigating transformations of exponential and logarithmic functions* Students should use graphing software to investigate the transformations of exponential and logarithmic functions. **Resource:** investigating-transformations.DOCX

NESA exemplar question* Explain how the graph of can be transformed to produce the graph of .

**Resource**: ma-e1-nesa-exemplar-questions-solutions.DOCX |       |       |
| Modelling and solving problems involving exponential and logarithmic functions (1 – 2 lessons) | * + interpret the meaning of the intercepts of an exponential graph and explain the circumstances in which these do not exist Critical and creative thinking icon
* establish and use the algebraic properties of exponential functions to simplify and solve problems (ACMMM064)
* solve problems involving exponential functions in a variety of practical and abstract contexts, using technology, and algebraically in simple cases (ACMMM067) AAM Paperclip icon  Information and communication technology capability icon
* model situations and solve simple equations involving logarithmic or exponential functions algebraically and graphically AAM
* identify contexts suitable for modelling by exponential and logarithmic functions and use these functions to solve practical problems (ACMMM066, ACMMM158) AAM
 | Solving problems involving exponential and logarithmic functions* Students should explicitly explore the interpretation of the meaning of intercepts in various problems.

NESA exemplar questions* The spread of a highly contagious virus can be modelled by the function , where is the number of days after the first case of sickness due to the virus has been diagnosed and is the total number of people who are infected by the virus in the first days. Find and interpret the meaning of , and . Use graphing software to find the graph of this function.
* In 2010, the city of Thagoras modelled the predicted population of the city using the equation . That year, the city introduced a policy to slow its population growth. The new predicted population was modelled using the equation . In both equations, is the predicted population and is the number of years after 2010. The graph shows the two predicted populations.

The graph shows two exponential functions that plot population against years.* + Use the graph to find the predicted population of Thagoras in 2030 if the population policy had NOT been introduced.
	+ In each of the two equations given, the value of is . What does represent?
	+ The guess-and-check method is to be used to find the value of , in .

Explain, with or without calculations, why is not a suitable first estimate for .With and , use the guess-and-check method and the equation to estimate the value of to two decimal places. Show at least TWO estimate values for , including calculations and conclusions.* + The city of Thagoras was aiming to have a population under in 2050. Does the model indicate that the city will achieve this aim? Justify your answer with suitable calculations.

**Resource**: ma-e1-nesa-exemplar-questions-solutions.DOCXConducting an experimentStudents to undertake a cooling investigation. This needs to occur after students have considered graphing and the transformation of exponential functions |       |       |

Reflection and evaluation

Please include feedback about the engagement of the students and the difficulty of the content included in this section. You may also refer to the sequencing of the lessons and the placement of the topic within the scope and sequence. All ICT, literacy, numeracy and group activities should be recorded in the ‘comments, feedback, additional resources used’ section.