 Sum of an arithmetic sequence

The activities below lead students to discover the sum of an arithmetic sequence.

Context: At the age of 7, Carl Friedrich Gauss is believed to have amazed his teachers by summing the integers from 1 to 100 almost instantly. How did he do this?

Activity 1: Find the sum of the first 100 integers.

Students to come up with methods of finding the sum of the first 100 integers.

Potential methods include:

1. Pairing the numbers: add 1 to 100, 2 to 99, 3 to 98 etc. to obtain $50×101=5050$
2. Pairing the numbers: add 1 to 99, 2 to 98, etc. to obtain $50×100+50=5050$ as the original term of 50 is not paired.

Discuss the first method in terms of $n$, $a$ and $l$.

Activity 2: Find the sum of the first 1000 integers.

Students to come up with methods of finding the sum of the first 1000 integers.

Potential methods include:

1. Pairing the numbers: add 1 to 1000, 2 to 999, 3 to 998 etc. to obtain $500×1001=500500$
2. Pairing the numbers: add 1 to 999, 2 to 998, etc. to obtain $500×1000+500=500500$ as the original term of 500 is not paired.

Discuss the first method in terms of $n$, $a$ and $l$.

Activity 3: Generalise the result.

Method 1:

* Consider the arithmetic series: $(1)$

$$S\_{n}=a+\left(a+d\right)+\left(a+2d\right)+\left(a+3d\right)+... +\left(a+\left(n-3\right)d\right)+\left(a+\left(n-2\right)d\right)+(a+\left(n-1\right)d)$$

* Consider the arithmetic series in reverse: $(2)$

$$S\_{n}=\left(a+\left(n-1\right)d\right)+\left(a+\left(n-2\right)d\right)+(a+\left(n-3\right)d+ … +\left(a+3d\right)+\left(a+2d\right)+\left(a+d\right)+a$$

* Add the two series together $(1)+(2)$

$$2S\_{n}=\left(a+\left(n-1\right)d\right)+\left(a+\left(n-1\right)d\right)+\left(a+\left(n-1\right)d\right)+... +\left(a+\left(n-1\right)d\right)+\left(a+\left(n-1\right)d\right)+\left(a+\left(n-1\right)d\right)$$

There are $n$ terms of $\left(a+\left(n-1\right)d\right)$

$$2S\_{n}=n\left[2a+\left(n-1\right)d\right]$$

$$S\_{n}=\frac{n}{2}\left[2a+\left(n-1\right)d\right]=\frac{n}{2}\left(a+l\right)$$

Method 2:

* Consider the arithmetic series: $(1)$

$$S\_{n}=a+\left(a+d\right)+\left(a+2d\right)+\left(a+3d\right)+... +\left(a+\left(n-2\right)d\right)+(a+\left(n-1\right)d)$$

* Consider the arithmetic series in reverse: $(2)$

$$S\_{n}=l+\left(l-d\right)+l-2d)+(l-3d)+…+ \left(l-\left(n-2\right)d\right)+(l-\left(n-1\right)d)$$

* Add the two series together $(1)+(2)$

$$S\_{n}=\left(a+l\right)+\left(a+l\right)+\left(a+l\right)+…+\left(a+l\right)+\left(a+l\right)$$

There are n terms of $\left(a+l\right)$

$$2S\_{n}=n\left(a+l\right)$$

$$S\_{n}=\frac{n}{2}\left(a+l\right)$$

$$S\_{n}=\frac{n}{2}\left(a+a+\left(n-1\right)d\right)$$

$$S\_{n}=\frac{n}{2}\left[2a+\left(n-1\right)d\right]$$